**Grading guide, Pricing Financial Assets, August 2015**

1. In the Black-Scholes-Merton Model the price, $p$, at time $t = 0$ of a European put option with strike $K$ on a stock with price $S_0$ at time $t = 0$ and expiry at $T > 0$ is given by:

$$p = K\mathsf{e}^{-rT}\Phi(-d_2) - S_0\Phi(-d_1)$$

where

$$d_1 = \frac{\ln S_0/K + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

and $\Phi$ is the standard Normal distribution function.

(a) What are the assumptions behind this model?

(b) Using the call-put-parity derive a formula for the pricing of a call option on the same stock and with same strike and expiry as the put.

(c) How can the put price formula be modified to cover the case of a dividend paying stock?

**Solution:**

(a) The discussion should at least cover

- the process that the underlying stock is assumed to follow, including the constant parameters of the Brownian Geometric Motion, and that no payments (dividends etc) are implicitly assumed for this pricing formula to be valid
- a constant risk free interest rate,
- that the put price is derived using an arbitrage equilibrium argument

(b) Given no dividends note that a portfolio of a long position in one call and a short in one put assuming no arbitrage is equivalent to a forward contract on the stock with forward price equal to the strike. With these assumptions the forward contract has a current value equal to the price of the stock less the discounted value of the strike. Thus the call price $c$ is

$$\begin{aligned} c =& p + S_0 - K\mathsf{e}^{-rT} \\ =& K\mathsf{e}^{-rT}\Phi(-d_2) - S_0\Phi(-d_1)) + S_0 - K\mathsf{e}^{-rT} \\ =& S_0(1 - \Phi(-d_1)) - K\mathsf{e}^{-rT}(1 - \Phi(-d_2)) \\ =& S_0\Phi(d_1) - K\mathsf{e}^{-rT}\Phi(d_2) \end{aligned}$$

(c) In Hull there are described two ways to include dividends in the analysis.

- You may consider the current stock price as consisting of two parts: The present value of dividends known to be paid until the expiry of the option and the remainder. The last part can be treated as the stock price in the formula for the European call. In principle this reduced stock price should have a volatility that is adjusted upwards if calculated from historical return data (but not if from consistent implicit volatilities of options with same expiry), Hull section 14.12.
- You may also consider dividends as a continuously paid yield $q$ (a special case of the above), Hull section 16.3, and substitute for $S_0$ in the above the reduced

$$S_0\mathsf{e}^{-qT}$$

2. (a) Define a Credit Default Swap (CDS) and describe the payment structure.

   (b) Let the probability of a borrower not defaulting at or before time $t$ be given by $V(t)$. What is the probability that the borrower will default between time $t$ and $t + \Delta t$ conditional on not being in default at time $t$? Use this to define the continuously compounded default hazard rate $\lambda(t)$.

   (c) How, and under which assumptions, may we estimate the hazard rate from a CDS spread $s$ on the relevant borrower? Under what probability measure would we say this estimate is derived? Compare this to a hazard rate that is derived from default frequencies and recovery rates published by a rating agency.

**Solution:**

(a)

**Definition 0.1** (Credit Default Swap)**.** *A **Credit Default Swap** (CDS) is a contract between a protection seller and a protection buyer based on a specified Nominal Principal. The protection buyer pays a running premium (the CDS spread) until maturity of the contract or a Credit Event on the Reference Entity of the CDS. At a Credit Event the protection seller pays a Specified Amount (Fixed or (1-Recovery) times the Nominal Principal) to the protection buyer*

(b) The conditional default probability from $t$ to $t + \Delta t$ is

$$\frac{V(t) - V(t + \Delta t)}{V(t)}$$

Given $V(t)$ the hazard rate is the rate of decay of survivors. Assuming differentiability you may define it as

$$\lambda(t) = -\frac{\frac{\partial V(t)}{\partial t}}{V(t)}$$

This can be motivated by considering the above discrete time step and letting the time step approach zero.

(c) Under a risk neutral probability measure the hazard rate may be derived directly from CDS-spreads if you make an appropriate assumption on the recovery rate $R$.

Assuming the CDS-spread is $s$ for a given maturity you can put

$$\bar{\lambda} = \frac{s}{1 - R}$$

if the recovery rate is known or the a market price for the recovery can be found. You may similarly bootstrap a hazard rate structure from a term structure of CDS spreads.

This analysis is conducted under a risk neutral ("$\mathbb{Q}$") measure, so that $(\lambda, R)$ will not be the same as frequencies and averages published by rating agencies, real world probabilities ("$\mathbb{P}$") (Hull section 23.4-5).

3. (a) In the Ho-Lee Model in continuous time the (instantaneous) short term interest rate $r$ is described by the process:

$$dr = \theta(t)dt + \sigma dz$$

   where $\theta(t)$ is a function of $t$, $\sigma$ is constant, and $dz$ the increment of a Brownian motion. What does this mean for the behaviour of the short term interest rate?

(b) Here as in other models of the short term interest rate $r$ the drift rate is made a function of calendar time. What is the purpose of the extra flexibility compared to the CIR (or Vasicek) type of models?

(c) In this model the solution for the price of a zero-coupon bond can be written

$$P(t,T) = A(t,T)\mathsf{e}^{-r(t)(T-t)}$$

Use Ito's lemma to derive the volatility of $P$ and comment on the result.

(d) Derive the duration of the bond.

**Solution:**

(a) Cf. Hull p. 690. It should be noted that the short term rates at any time are normally distributed with a constant volatility, but time-varying mean. In particular it does not rule out negative interest rates.

(b) Cf. Hull p.689ff. The time-dependent drift term in these models is introduced to be able to incorporate a given, initial term structure, making the the values derived from it "arbitrage-free" in relation to the existing securities priced on the current term structure (assuming these to be arbitrage free). This is in contrast to the CIR and Vasicek models of the "Equilibrium"-type that put restrictions on the possible initial term structure (as it is a function with constant parameters and of the current spot rate only).

(c) We note that

$$\frac{\partial P}{\partial r} = -(T-t)P$$

Thus the volatility term process for $dP$ is $(T-t)\sigma P$ (as the sign does not matter).

(d) The answer depends on the definition of duration applied. The version in Hull (p.687) is

$$D = -\frac{\frac{\partial P}{\partial r}}{P} = (T-t)$$